Chad's Self-Help Homology Tutorial For The Simple(x)-Minded

A full-color Extravaganza

With very sincere thanks and apologies to Lori Ziegelmeier and Tom Halverson, who actually know topology and tried to explain it to me.

Main idea: You have a set of data points and you want to study its structure. But like me, you only know linear algebra (on a good day).

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I. Simplices + simplicial complexes

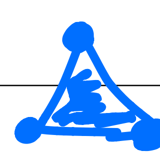
- Start with your data pts. Easiest to think of data in $\mathbb{R}^n$ but chose whatevs space you want

- Each data pt. is a 0-simplex

- Now we build your data into a structure called a simplicial complex

- Many ways to do this, but we use the Vietoris-Rips complex
More on this at the very end

- Choose a distance whereby 2 pts within that distance get connected with an edge. An edge is a 1-simplex 

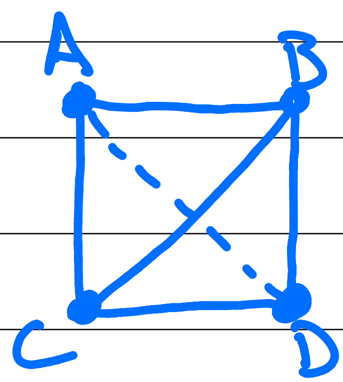
- Three pairwise-connected points form a filled-in triangle which is a 2-simplex 

- Four pairwise-connected points form a filled-in tetrahedron. This is a 3-simplex. I won't sketch it because I suck at art even more than topology.

- And so on.

- Now you built a big (higher-dimensional?) object out of your data!

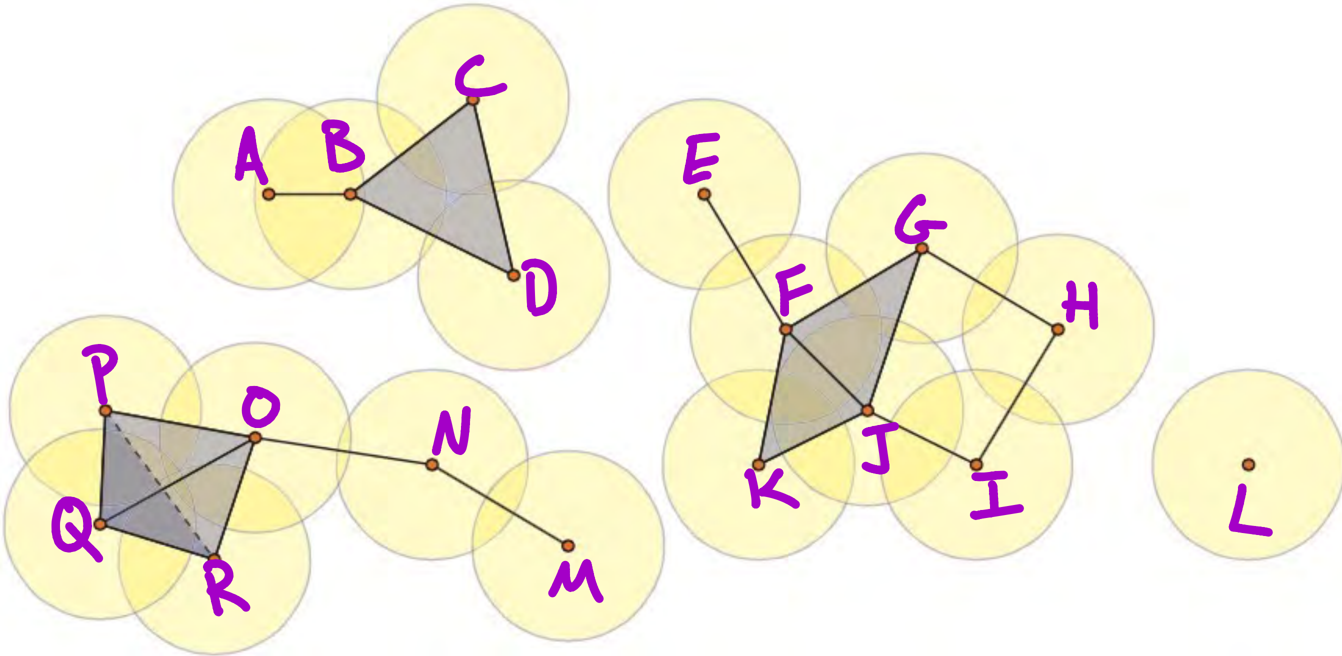
ex.



(If you are a better artist than me you can do shading to make it more tetrahedroney)

0-simplices: [A], [B], [C], [D]
1-simplices: [AB], [BD], [DC], [CA], [AD], [BC]
2-simplices: [ABC], [ABD], [BDC], [ADC]
3-simplex: [ABDC]

Exercise 1: Name simplices in this complex:



By the way, simplices have an orientation. An odd permutation of the simplex negates it.

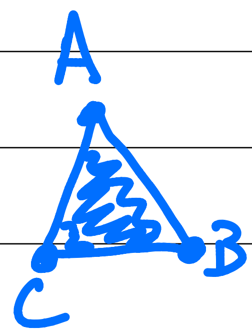
ex. $[NM] = -[MN]$

II. Boundaries

The boundary of a simplex is some simplices of 1 lower dimension. There is a boundary operator ∂_k that does this. We use for the subscript k the dimension of the simplex whose boundary we want. The formula is

$$\partial_k[v_0, \dots, v_k] = \sum_{i=0}^{k} (-1)^i \underbrace{[v_0, \dots, v_k]}_{\text{but exclude the } i^{\text{th}} \text{ vertex}}$$

ex.



$$\partial_2[ABC] = +[BC] - [AC] + [AB]$$
$$= +[BC] + [CA] + [AB]$$

$$\partial_1[BC] = C - B$$

$$\partial_0[B] = 0$$

Exercise 2: Find the boundary of the tetrahedron [ABCD]
Then show that the boundary has no boundary! (This is always true)

III k-chains and k-cycles

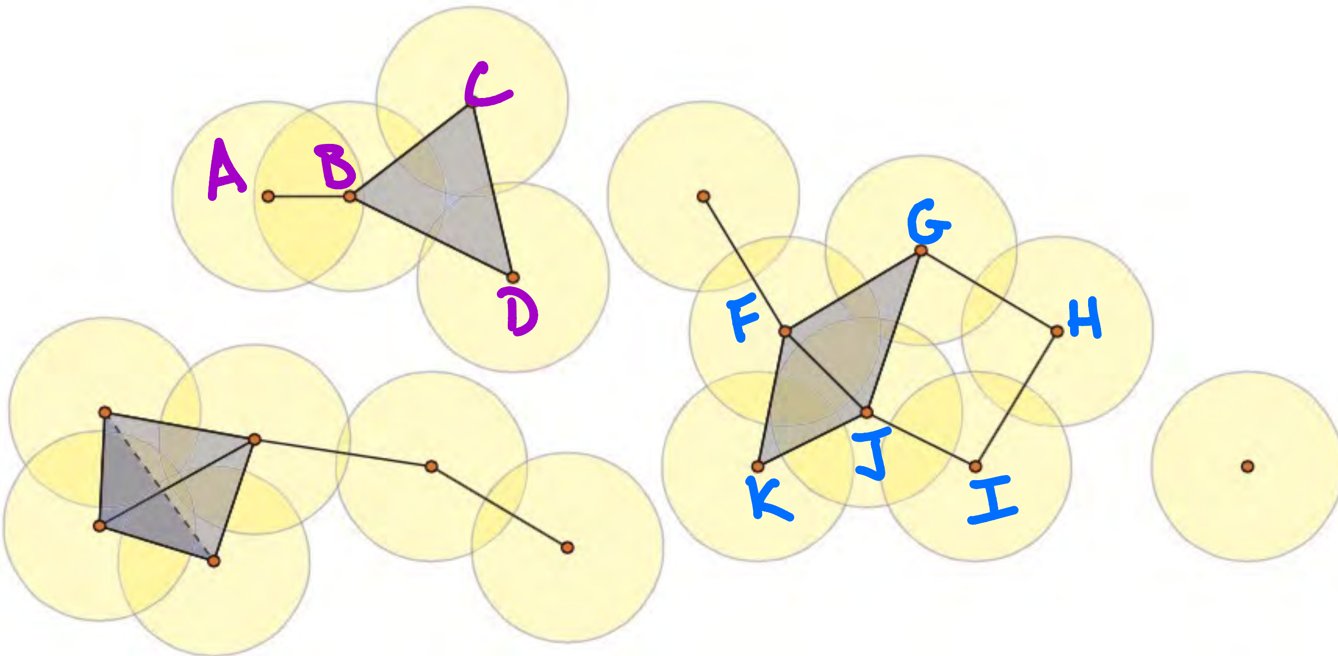
A k-chain is just a linear combination of k-simplices.

ex. If we have 2-simplices [ABC], [ABD], [ACD], [BCD], then [ABD] + [BCD] is a 2-chain

ex. If we have 1-simplices [AB], [BC], [CD], [AD], then [AB] + [BC] + [AD] is a 1-chain

A k-cycle is just a k-chain that has a boundary of 0.

ex. Consider the simplicial complex



The 1-chain:

$[FG]+[GH]+[HI]+[IJ]+[JK]+[KF]$

is a 1-cycle because its boundary is

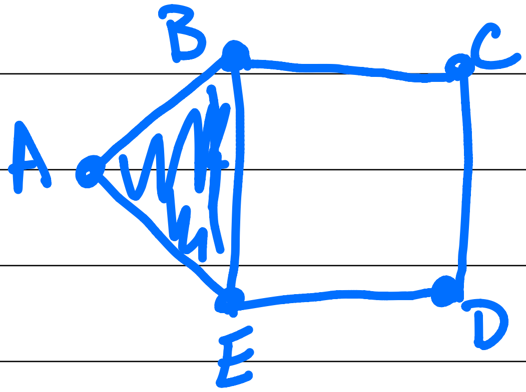
$G-F+H-G+I-H+J-I+K-J+F-K=0$

Exercise 3: Is $[AB]+[BC]+[CD]+[DB]$ a cycle?

IV Homology

We say that two cycles are homologous if they differ by a boundary.

<u>ex.</u> Consider the simplicial complex



Let $X = [AB] + [BC] + [CD] + [DE] + [EA]$

Let $Y = [BC] + [CD] + [DE] + [EB]$

(You can check that the 1-chains X and Y are 1-cycles)

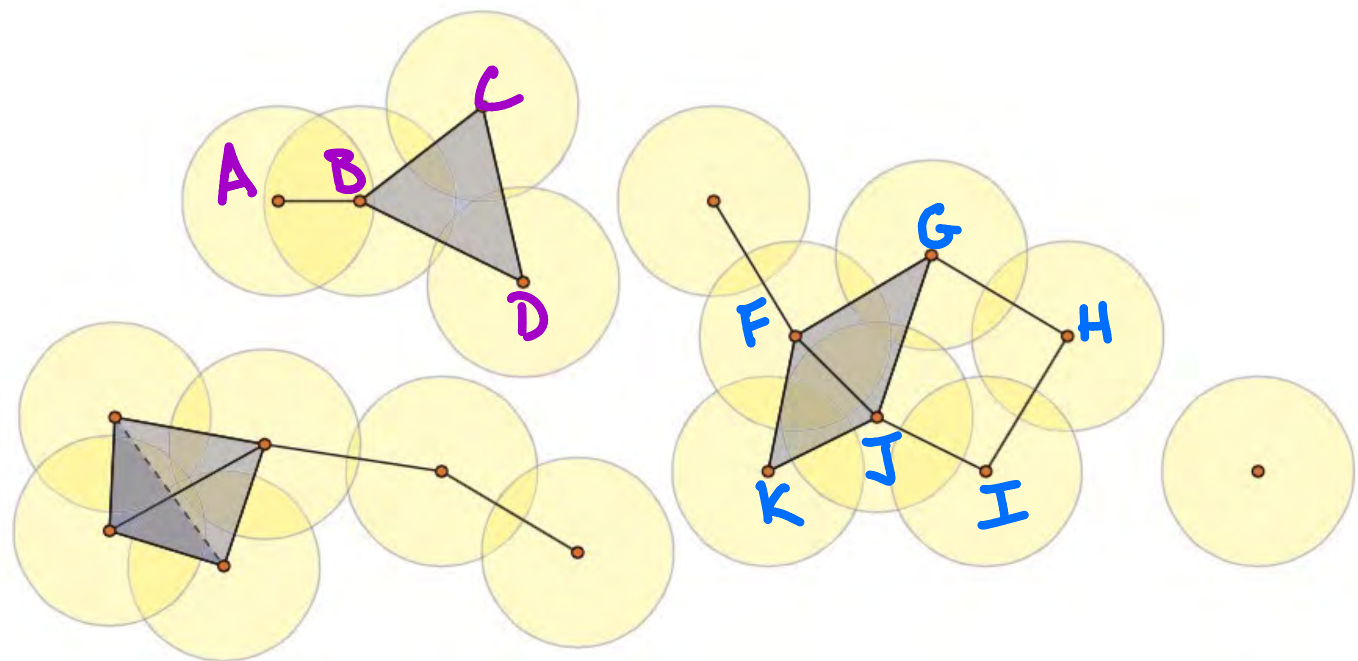
Note $X - Y = [AB] - [AE] + [BE]$

$$= [BE] - [AE] + [AB]$$

$$= \partial_2 [ABE]$$

So X and Y are homologous.

Exercise 4:



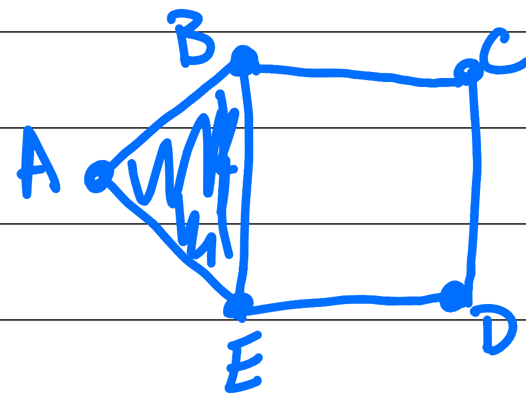
Let $X = [FG] + [GJ] + [JK] + [KF]$

Let $Y = [FJ] + [JK] + [KF]$

Are X + Y homologous?

As you will see later, we will want to figure out how many homologically distinct cycles there are and discard the ones that are boundaries.

Recall this example:



There are three 1-cycles, namely

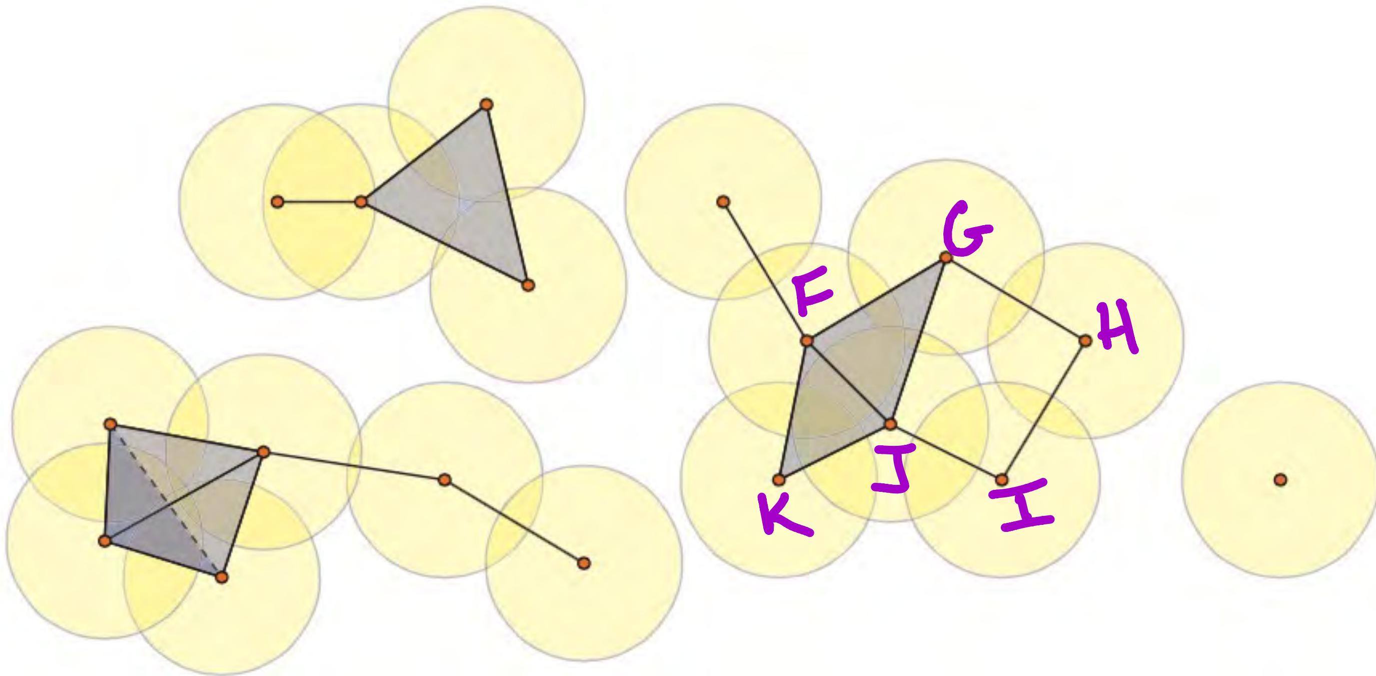
$X = [AB] + [BC] + [CD] + [DE] + [EA]$

$Y = [BC] + [CD] + [DE] + [EB]$

$Z = [AB] + [BE] + [EA]$

We have already shown that X + Y are homologous, and $Z = X - Y$ is a boundary. We lump together X + Y and discard Z, giving us one remaining homology class.

Exercise 5:



In the labeled subcomplex, how many homologically distinct 1-cycles are there after discarding boundary classes?

V. Betti #'s

Betti #'s are topological invariants that measure the number of k-dimensional holes in an object.

The fact that they are topological invariants means that they don't change if you warp/stretch an object (without tearing it). If I told you that a mystery object has certain Betti #'s, it wouldn't necessarily tell you everything about the topology of the object, but it would still tell you a lot!

More specifically, the Betti #'s b_k tell you the following about an object:

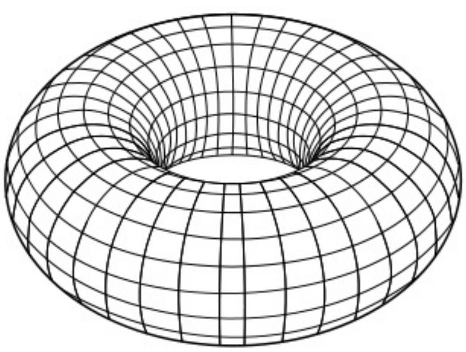
b_0 - # of connected components

b_1 - # of topological circles (or "holes")

b_2 - # of trapped 3-d volumes

and so on.

For instance, a two-torus has Betti #'s



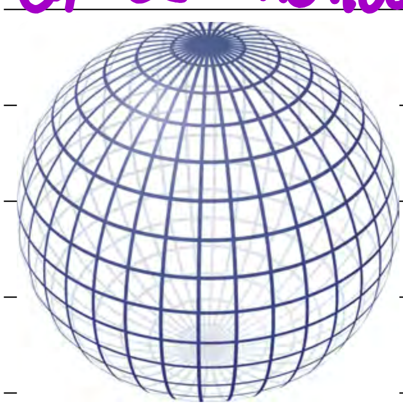
$b_0 = 1$ $\quad$ $b_3 = 0$

$b_1 = 2$ $\quad$ $b_4 = 0$

$b_2 = 1$ $\quad$ etc.

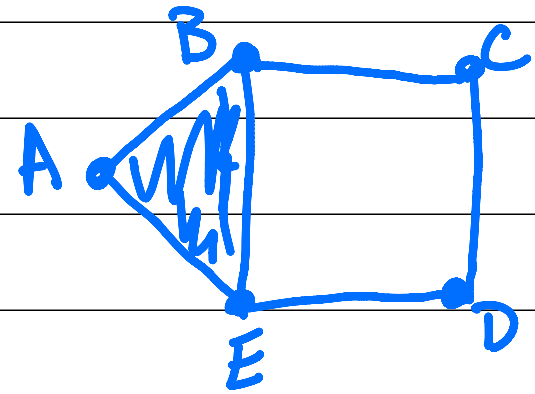
Sometimes, we write as a vector, $b = (1, 2, 1, 0, \dots)$

Exercise 6: What are the Betti #'s of a (hollow) sphere?



Now the hand-wavey magic:
For our simplicial complexes,
b_k = # of homologically distinct k-cycles after discarding k-boundaries

Recall this example:



We showed earlier that there is 1 class of 1-cycles after discarding boundaries. Therefore, $b_1 = \underline{1}$. This looks right -- there is one hole in the complex.

BUT CHAD (you complain) THIS IS A PAIN IN THE @$$ BECAUSE COUNTING ALL THIS STUFF TAKES FOREVER. :(

That is why we have linear algebra.

VI. Calculating Betti #'s of simplicial complexes using linear algebra.

Consider chains going from higher dimensions to lower, as mapped by boundary operators. For concreteness,

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

To get b_1, we are supposed to find homologically distinct 1-cycles and throw away 1-cycles that are boundaries.

Well, 1-cycles by definition have a boundary of 0, so they are in the kernel of ∂_1. 1-cycles that are themselves boundaries of something are in the image of ∂_2. So... *MAGIC*

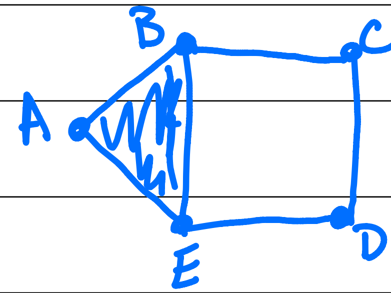
$b_1 = \dim \operatorname{Ker} \partial_1 - \dim \operatorname{Im} \partial_2$ or in general,

$b_k = \dim \operatorname{Ker} \partial_k - \dim \operatorname{Im} \partial_{k+1}$

Apparently this has something to do with quotient spaces but I only took algebra as an undergrad and I was a terrible student who never paid attention.

Now all we have to do is put our boundary operators in matrix form (once we choose coordinates).

Example: My fave!



First, do

$$C_2 \xrightarrow{\partial_2} C_1$$

List all 2-simplices

List all 1-simplices

$$\begin{array}{c|c} & ABE \\ \hline AB & \\ BC & \\ CD & \\ DE & \\ AE & \\ BE & \end{array}\;\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

why? because

$$\partial_2[ABE] = [BE] - [AE] + [AB]$$

So this is a matrix version of ∂_2

Now do ∂_1.

$$C_1 \xrightarrow{\partial_1} C_0$$

$$\begin{array}{c c} & \overbrace{\begin{array}{cccccc} AB & BC & CD & DE & AE & BE \end{array}}^{\text{List all 1-simplices}} \\ \text{List all 0-simplices}\left\{\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}\right. & \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

As an example, $\partial_1[AE] = [E] - [A]$

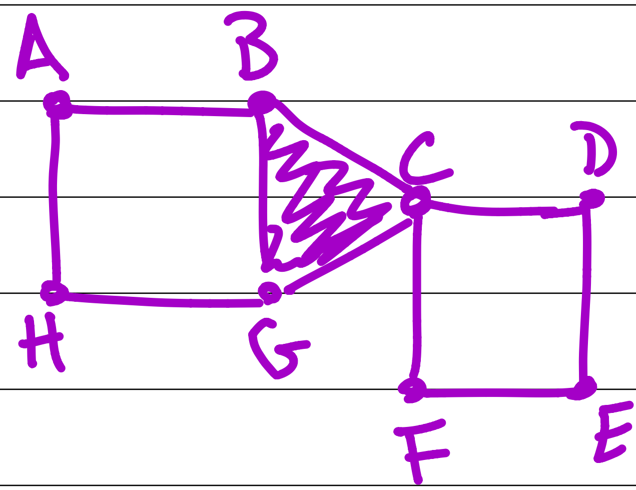
Now compute by hand or use Mathematica.

$\dim \operatorname{Ker} \partial_1 = 2$

$\dim \operatorname{Im} \partial_2 = 1$

So $b_1 = 2 - 1 = 1$ 1 hole!!

Exercise 7: Consider this simplicial complex. Do the linear algebra to calculate b_1.



VII. Shut. Up. Already.

OK, almost there! I just want to remind you that we started with data, built a simplicial complex, and characterized its topological structure by calculating Betti #'s.

In a lot of examples I focused on Betti # b_1 for concreteness, but the ideas work for b_0 or b_2 or $b_{whatever}$.

I find it amazing that it all boils down to a linear algebra computation. We did the matrix stuff by hand but there are great software packages to automate it.

Remember way back at the start when you built a simplicial complex by connecting points within a certain distance? That choice of distance is (probably) arbitrary. A natural thing to do is to consider many different connection distances and study the topology of all the resulting complexes. The topological features that stick around over a large range of connection distances are said to be *persistent*.
Now you are talking about persistent homology, perhaps the subject of a future whimsical tutorial.

Finally ... if *I* can do it, *you* can do it!

THE END

And rember: if you don't love topology, how in the hell r u gonna love yourself?