

Chad's Self-Help Homology Tutorial For The Simple(x)-Minded

A full-color Extravaganza




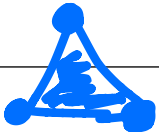
With very sincere thanks and
apologies to Lori Ziegelmeier and
Tom Halverson, who actually know
topology and tried to explain it
to me.



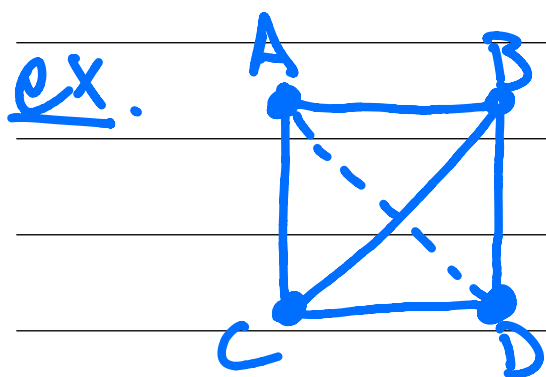
Main idea: You have a set of data
points and you want to study its
structure. But like me, you only
know linear algebra (on a good day).

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I. Simplices + simplicial complexes

- Start with your data pts. *Easiest to think of data in \mathbb{R}^n but choose whatever space you want*
- Each data pt. is a 0-simplex
- Now we build your data into a structure called a simplicial complex
- Many ways to do this, but we use the Vietoris-Rips complex
More on this at the very end
- Choose a distance whereby 2 pts within that distance get connected with an edge. An edge is a 1-simplex 
- Three pairwise-connected points form a filled-in triangle which is a 2-simplex 

- Four pairwise-connected points form a filled-in tetrahedron. This is a 3-simplex. I won't sketch it because I suck at art even more than topology.
- And so on.
- Now you built a big (higher-dimensional?) object out of your data!



(If you are a better artist than me you can do shading to make it more tetrahedrony)

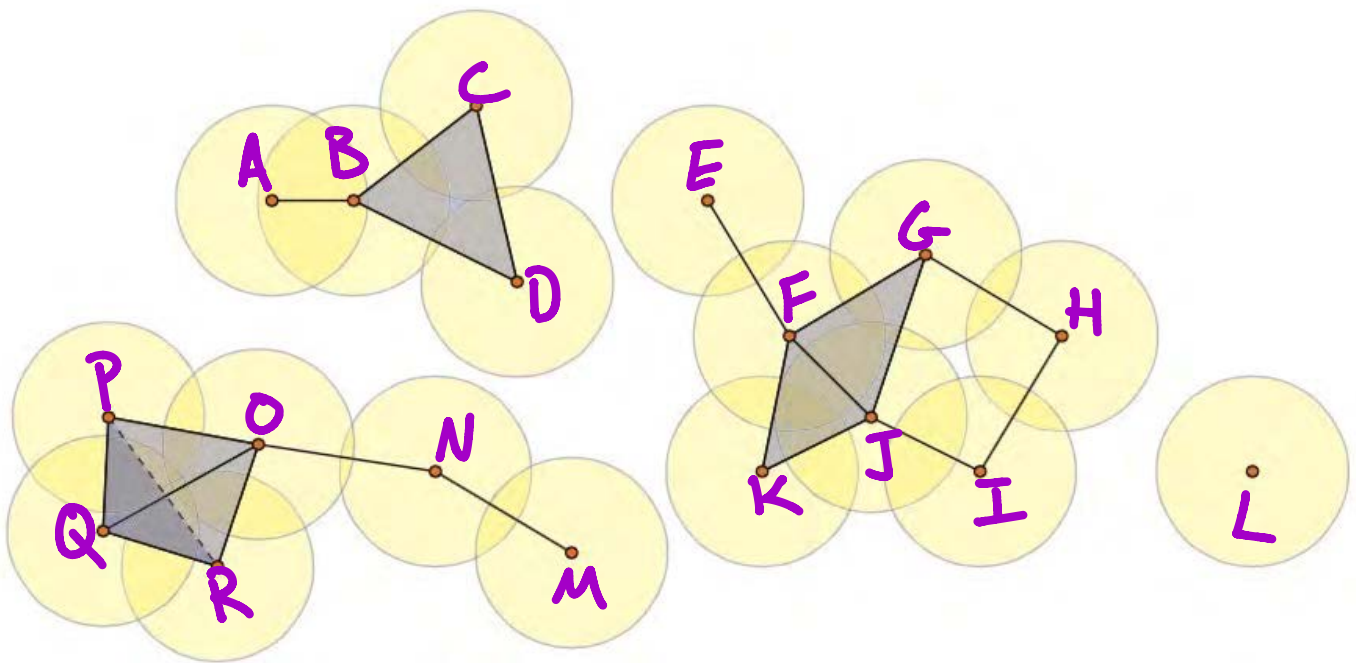
0-simplices: $[A], [B], [C], [D]$

1-simplices: $[AB], [BD], [DC], [CA],$
 $[AD], [BC]$

2-simplices: $[ABC], [ABD], [BDC], [ADC]$

3-simplex: $[ABDC]$

Exercise 1: Name simplices in this complex:



0-simplices: $[A], [B], \dots, [Q]$

1-simplices: $[AB], [BC], \dots$

2-simplices: $[BCD], [FGJ], [FJK],$
 $[POR], [ORQ], [RQP], [QPO]$

3-simplex: $[PORQ]$

By the way, simplices have an orientation.
An odd permutation of the simplex
negates it.

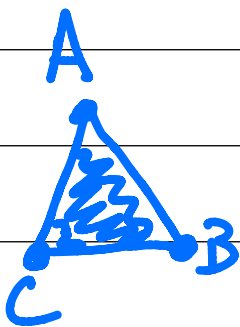
ex. $[NM] = -[MN]$

II. Boundaries

The boundary of a simplex is some simplices of 1 lower dimension. There is a boundary operator ∂_k that does this. We use for the subscript k the dimension of the simplex whose boundary we want. The formula is

$$\partial_k [v_0, \dots, v_k] = \sum_{i=0}^k (-1)^i \underbrace{[v_0, \dots, v_k]}_{\text{but exclude the } i^{\text{th}} \text{ vertex}}$$

ex.



$$\begin{aligned}\partial_2 [ABC] &= +[BC] - [AC] + [AB] \\ &= +[BC] + [CA] + [AB]\end{aligned}$$

$$\partial_1 [BC] = C - B$$

$$\partial_0 [B] = 0$$

Exercise 2: Find the boundary of the tetrahedron $[ABCD]$

Then show that the boundary has no boundary! (This is always true)

$$\partial_3[ABCD] =$$

$$[BCD] - [ACD] + [ABD] - [ABC]$$

$$\text{Now take } \partial_2 \partial_3[ABCD] =$$

$$[CD] - [BD] + [BC]$$

$$- [CD]$$

$$+ [BD]$$

$$- [BC]$$

$$+ [AD] - [AC]$$

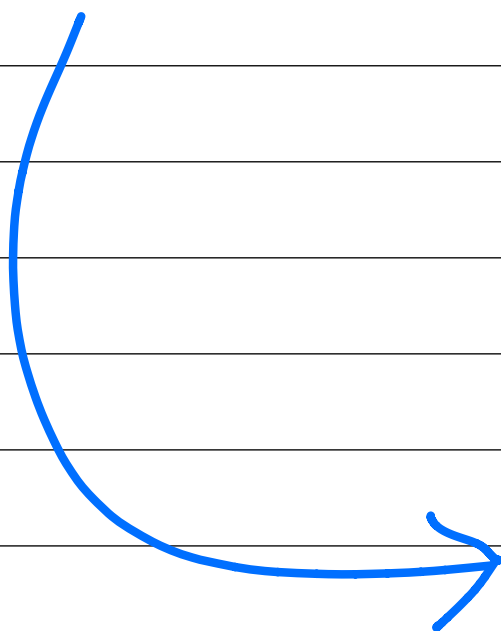
$$- [AD]$$

$$+ [AC]$$

$$+ [AB]$$

$$- [AB]$$

$$= 0!$$



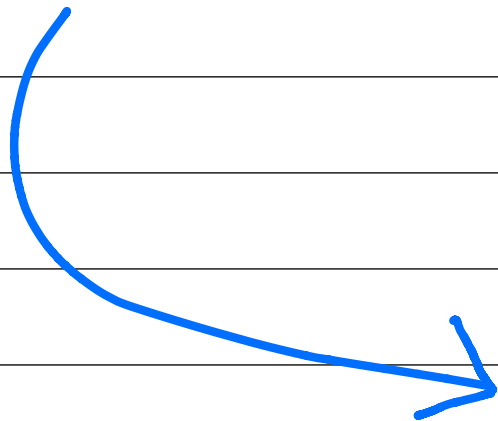
III k -chains and k -cycles

A k -chain is just a linear combination of k -simplices.

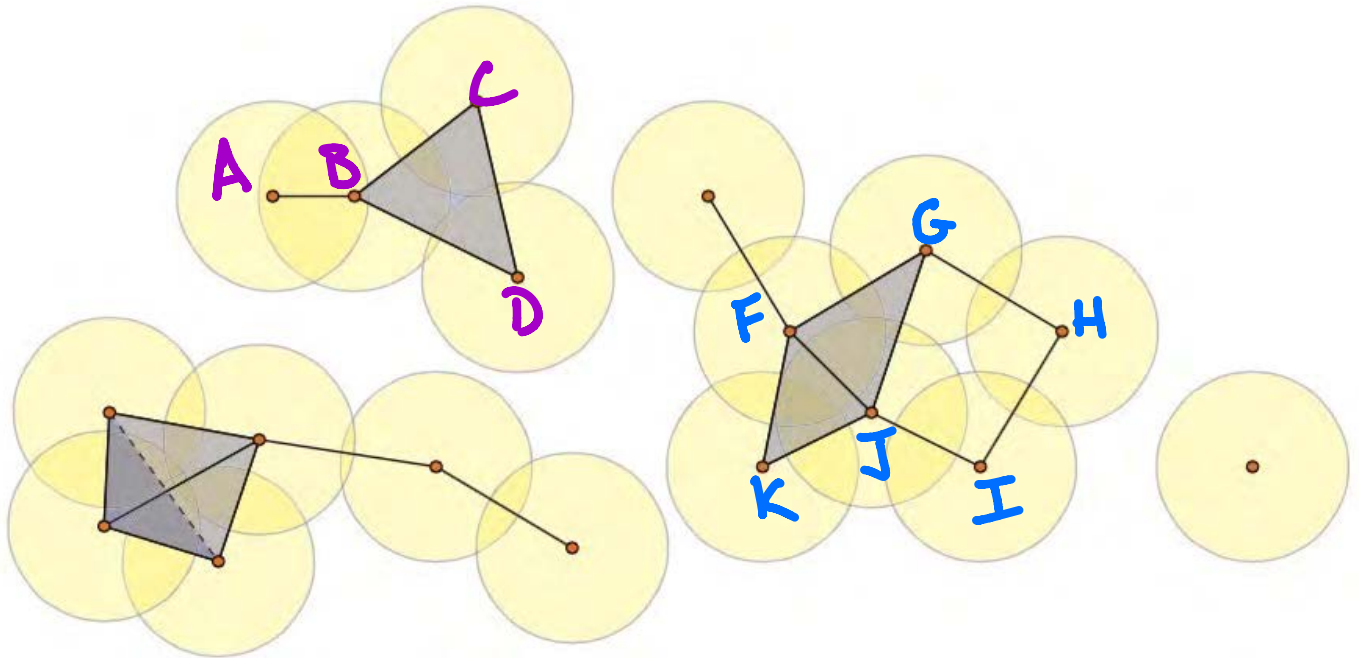
ex. If we have 2-simplices $[ABC], [ABD], [ACD], [BCD]$, then $[ABD] + [BCD]$ is a 2-chain

ex. If we have 1-simplices $[AB], [BC], [CD], [AD]$, then $[AB] + [BC] + [AD]$ is a 1-chain

A k -cycle is just a k -chain that has a boundary of 0.



ex. Consider the simplicial complex



The 1-chain:

$[FG] + [GH] + [HI] + [IJ] + [JK] + [KF]$
is a 1-cycle because its boundary is
 $G - F + H - G + I - H + J - I + K - J + F - K = 0$

Exercise 3: Is $[AB] + [BC] + [CD] + [DB]$
a cycle?

Applying ∂_1 , we have

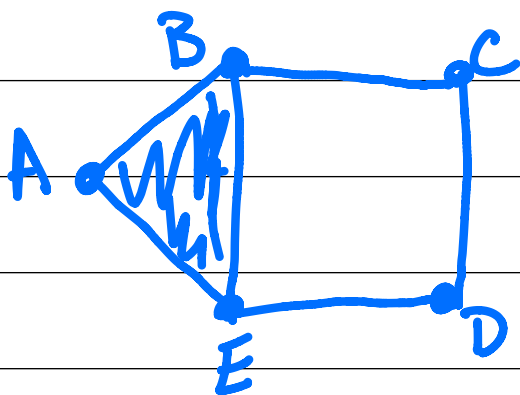
$$B - A + C - B + D - C + B - D = -A + B \neq 0$$

Not a cycle.

IV Homology

We say that two cycles are homologous if they differ by a boundary.

ex. Consider the simplicial complex



$$\text{Let } X = [AB] + [BC] + [CD] + [DE] + [EA]$$

$$\text{Let } Y = [BC] + [CD] + [DE] + [EB]$$

(You can check that the 1-chains X and Y are 1-cycles)

$$\text{Note } X - Y = [AB] - [AE] + [BE]$$

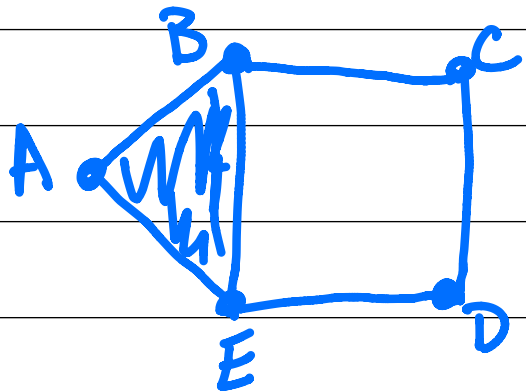
$$= [BE] - [AE] + [AB]$$

$$= \partial_2 [ABE]$$

So X and Y are homologous.

As you will see later, we will want to figure out how many homologically distinct cycles there are and discard the ones that are boundaries.

Recall this example:

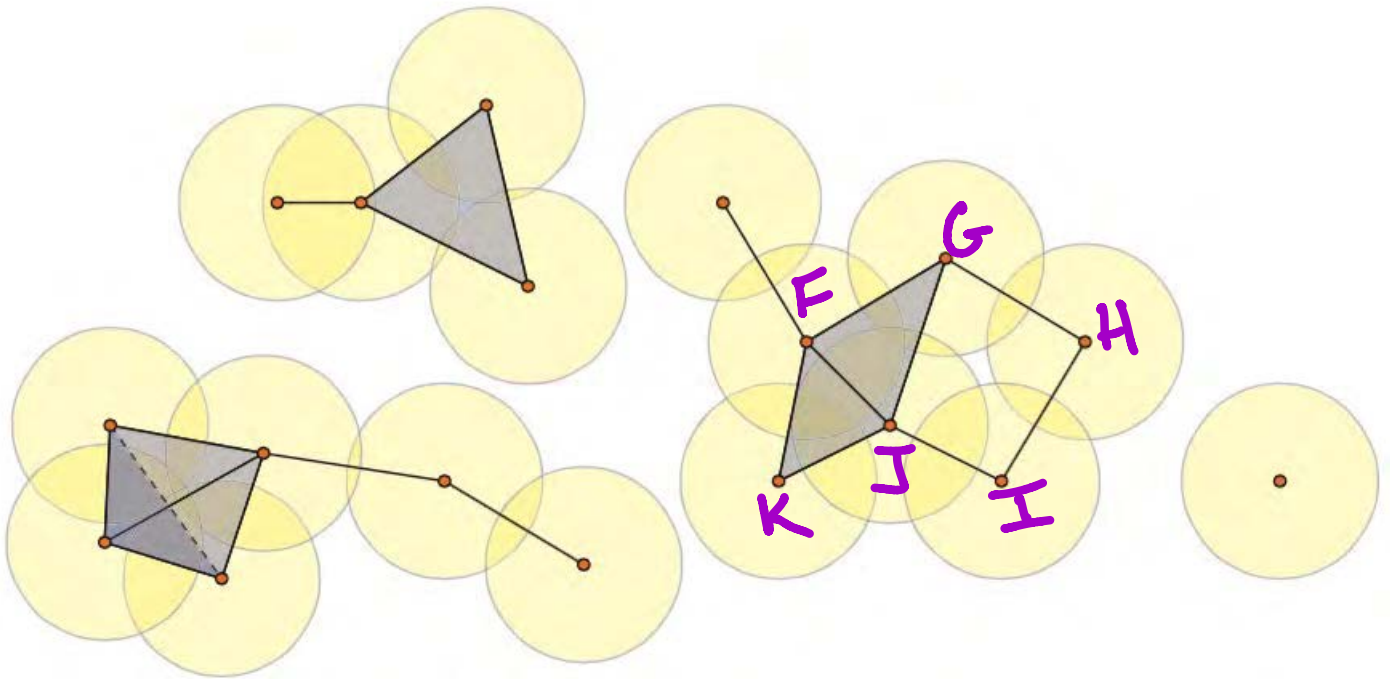


There are three 1-cycles, namely

$$X = [AB] + [BC] + [CD] + [DE] + [EA]$$
$$Y = [BC] + [CD] + [DE] + [EB]$$
$$Z = [AB] + [BE] + [EA]$$

We have already shown that $X + Y$ are homologous, and $Z = X - Y$ is a boundary. We lump together $X + Y$ and discard Z , giving us one remaining homology class.

Exercise 5:



In the labeled subcomplex, how many homologically distinct 1-cycles are there after discarding boundary classes?

First, we enumerate the 1-cycles

$$C_1 = [KF] + [FJ] + [JK]$$

$$C_2 = [FG] + [GH] + [HF]$$

$$C_3 = [KF] + [FG] + [GH] + [JK]$$

$$C_4 = [GH] + [HI] + [IJ] + [JG]$$

$$C_5 = [FG] + [GH] + [HI] + [IJ] + [JF]$$

$$C_6 = [KF] + [FG] + [GH] + [HI] + [IJ] + [JK]$$

First, let's deal with $C_{1,2,3}$

Write C_1 as $[KF] - [JF] + [JK]$

$$= \partial_2 [JKF] \rightarrow \text{discard } C_1$$

Write C_2 as $[FG] - [JG] + [JF]$

$$= \partial_2 [JFG] \rightarrow \text{discard } C_2$$

Note $C_3 - C_2 = [KF] + [JK] - [JF]$

$$= \partial_2 [JKF]$$

Since $C_3 + C_2$ differ by a boundary, lump together. Since we discarded

C_2 , we discard C_3 .

Now deal with $C_{3,4,5}$, all not boundaries.

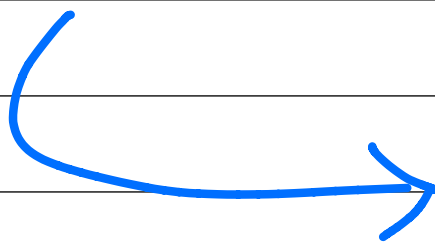
Note $C_6 - C_5 = [KF] + [JK] - [JF]$

$$= \partial_2 [JKF] \rightarrow \text{lump } C_5 + C_6 \text{ together}$$

Note $C_5 - C_4 = [FG] + [JF] - [JG]$

$$= \partial_2 [JFG] \rightarrow \text{lump } C_4, C_5, C_6 \text{ together}$$

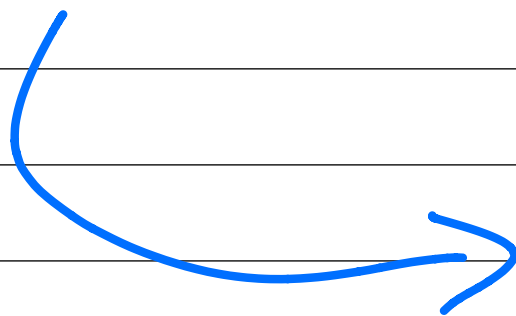
So, there is 1 distinct class that we hold on to.



V. Betti #'s

Betti #'s are topological invariants that measure the number of K -dimensional holes in an object.

The fact that they are topological invariants means that they don't change if you warp/stretch an object (without tearing it). If I told you that a mystery object has certain Betti #'s, it wouldn't necessarily tell you everything about the topology of the object, but it would still tell you a lot!



More specifically, the Betti #'s b_k tell you the following about an object:

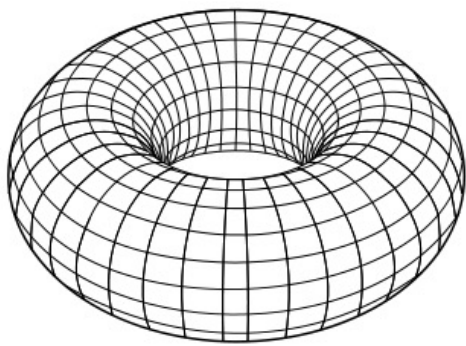
b_0 - # of connected components

b_1 - # of topological circles (or "holes")

b_2 - # of trapped 3-d volumes

and so on.

For instance, a two-torus has Betti #'s



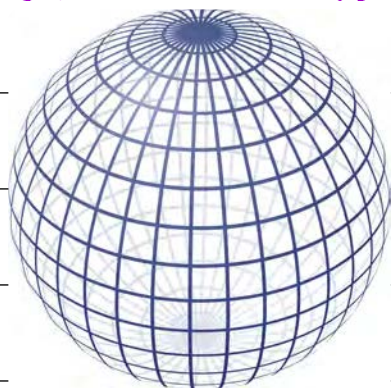
$$b_0 = 1 \quad b_3 = 0$$

$$b_1 = 2 \quad b_4 = 0$$

$$b_2 = 1 \quad \text{etc.}$$

Sometimes, we write as a vector, $b = (1, 2, 1, 0, \dots)$

Exercise 6: What are the Betti #'s of a (hollow) sphere?



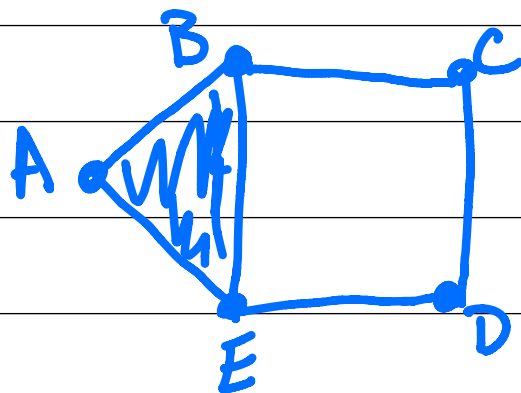
$$b = (1, 0, 1, 0, \dots)$$

Now the hard-wavy magic:

For our simplicial complexes,
 $b_k = \#$ of homologically distinct k -cycles
after discarding k -boundaries

Recall this example:

We showed earlier
that there is



1 class of 1-cycles

after discarding boundaries. Therefore,
 $b_1 = 1$. This looks right -- there is
one hole in the complex.

BUT CHAD (you complain) THIS IS
A PAIN IN THE @\$\$ BECAUSE
COUNTING ALL THIS STUFF
TAKES FOREVER. ;)

That is why we have linear algebra.

VI. Calculating Betti #'s of simplicial complexes using linear algebra.

Consider chains going from higher dimensions to lower, as mapped by boundary operators. For concreteness,

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

To get b_1 , we are supposed to find homologically distinct 1-cycles and throw away 1-cycles that are boundaries.

Well, 1-cycles by definition have a boundary of 0, so they are in the kernel of ∂_1 . 1-cycles that are themselves boundaries of something are in the image of ∂_2 . So... ***MAGIC***

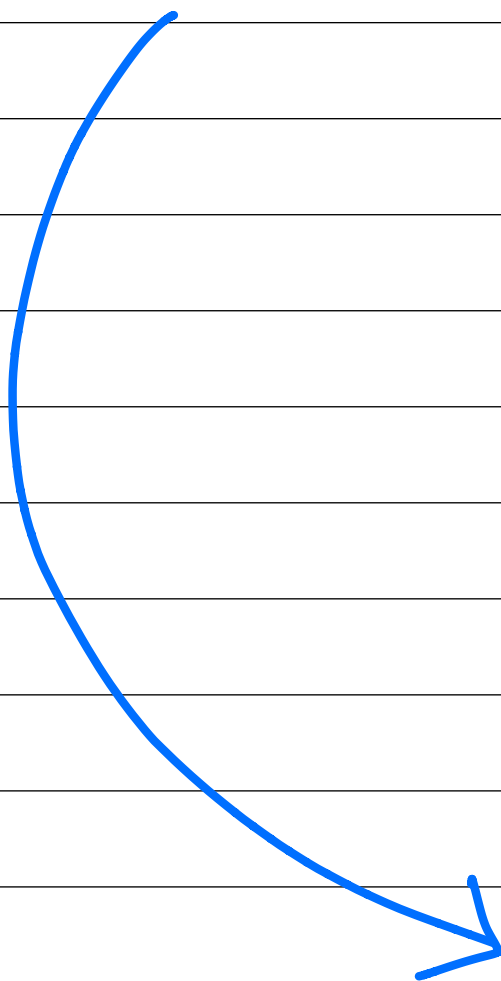
$$b_1 = \dim \text{Ker } \partial_1 - \dim \text{Im } \partial_2$$

or in general,

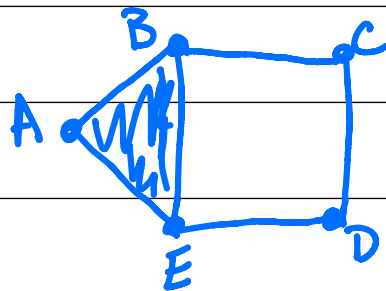
$$b_k = \dim \text{Ker } \partial_k - \dim \text{Im } \partial_{k+1}$$

Apparently this has something to do with quotient spaces but I only took algebra as an undergrad and I was a terrible student who never paid attention.

Now all we have to do is put our boundary operators in matrix form (once we choose coordinates).



Example: My save!



First, do

$$C_2 \xrightarrow{\partial_2} C_1$$

List all
2-simplices

List all 1-simplices

	ABE
AB	1
BC	0
CD	0
DE	0
AE	-1
BE	1

why? because
 $\partial_2 [AB\bar{E}] = [BE] - [AE] + [AB]$

So this is a
matrix version
of ∂_2

Now do ∂_1 .

$$C_1 \xrightarrow{\partial_1} C_0$$

List all 1-simplices

		AB	BC	CD	DE	AE	BE
List all 0-simplices	A	-1	0	0	0	-1	0
	B	1	-1	0	0	0	-1
	C	0	1	-1	0	0	0
	D	0	0	1	-1	0	0
	E	0	0	0	1	1	1

As an example, $\partial_1[AE] = [E] - [A]$

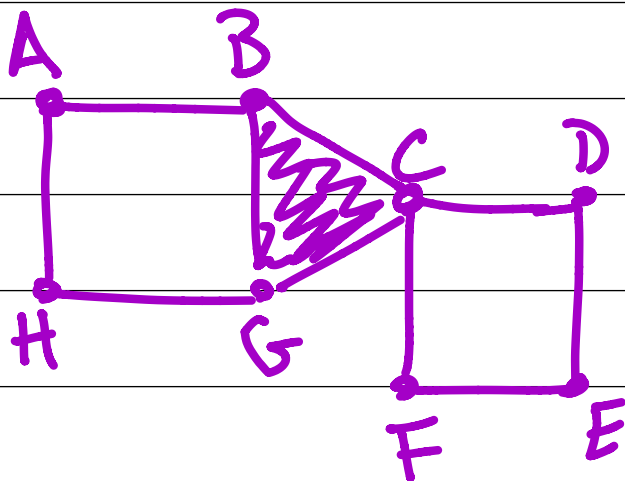
Now compute by hand or use Mathematica.

$$\dim \text{Ker } \partial_1 = 2$$

$$\dim \text{Im } \partial_2 = 1$$

$$\text{So } b_1 = 2 - 1 = 1 \quad \text{1 hole!!}$$

Exercise 7: Consider this simplicial complex. Do the linear algebra to calculate b_1 .



For ∂_2 :

$$\begin{array}{l}
 \text{BCG} \\
 \begin{bmatrix}
 0 \\
 1 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 1 \\
 0 \\
 0 \\
 -1
 \end{bmatrix}
 \end{array}$$

For ∂_1 :

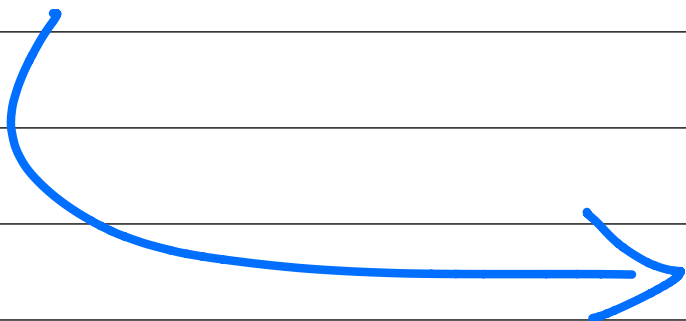
	AB	BC	CD	DE	EF	FC	CG	GH	HA	BG
A	-1	0	0	0	0	0	0	0	1	0
B	1	-1	0	0	0	0	0	0	0	-1
C	0	1	-1	0	0	1	-1	0	0	0
D	0	0	1	-1	0	0	0	0	0	0
E	0	0	0	1	-1	0	0	0	0	0
F	0	0	0	0	1	-1	0	0	0	0
G	0	0	0	0	0	0	1	-1	0	1
H	0	0	0	0	0	0	0	1	-1	0

Using Mathematica,

$$\dim \text{Ker } \partial_1 = 3$$

$$\dim \text{Im } \partial_2 = 1$$

$$\text{So } b_1 = 3 - 1 = 2$$



VII. Shut. Up. Already.

OK, almost there! I just want to remind you that we started with data, built a simplicial complex, and characterized its topological structure by calculating Betti #'s.

In a lot of examples I focused on Betti # b_1 for concreteness, but the ideas work for b_0 or b_2 or b_{whatever} .

I find it amazing that it all boils down to a linear algebra computation. We did the matrix stuff by hand but there are great software packages to automate it.

Remember way back at the start when you built a simplicial complex by connecting points within a certain distance? That choice of distance is (probably) arbitrary. A natural thing to do is to consider many different connection distances and study the topology of all the resulting complexes. The topological features that stick around over a large range of connection distances are said to be persistent.

Now you are talking about persistent homology, perhaps the subject of a future whimsical tutorial.

Finally ... if I can do it, you can do it!

♥♥ THE
END



And remember: if you don't love topology, how in the hell r u gonna love yourself?